JOURNAL OF APPROXIMATION THEORY 34, 91–96 (1982)

Optimal Subspaces for n-Widths of p-Ellipsoids

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Communicated by Richard S. Varga

Received December 1, 1980

The n-width of a compact set K in a finite or infinite dimensional Banach space B is defined after Kolmogorov [4, 7] as

$$d_n(K, B) = \inf_{L, \dim L \leqslant n} \sup_{x \in K} \operatorname{dist}(x, L),$$

where

$$\operatorname{dist}(x, M) = \inf_{y \in M} \|x - y\|_B.$$

The computation of asymptotics of these characteristics for concrete classes of compacta and Banach spaces of smooth and holomorphic functions was one of the central problems of the theory of approximation for the last two decades. However, except for Hilbert spaces, it seems there is no information on the structure of the set $\mathscr{L}_n$ of those n-dimensional subspaces in B which give the minimal value of the function

$$\delta_n(K; L) = \sup_{x \in K} \operatorname{dist}(x, L), \qquad \dim L = n.$$

Karlovitz [2, 3] and Ismagilov and Mityagin (see [1, Theorem 1.2]) give the complete description of the set in the case of an ellipsoid in a Hilbert space. Melkman and Micchelli [5] showed that spline subspaces are optimal in Sobolev spaces W_2^r (one variable).

In this note we extend the Ismagilov–Mityagin analysis ($p = 2$) to the case of p-ellipsoids

$$\mathscr{E} = \left\{ x \in l_p^N \colon \sum_{i=0}^{N} \left| \frac{x_i}{a_i} \right|^p \leqslant 1 \right\}$$

for any p, $1 \leqslant p \leqslant \infty$. Assume that $\{a_i\}_{i=0}^N$, $N \leqslant \infty$, is a strictly decreasing sequence of positive scalars.

0021-9045/82/010091–06$02.00/0

It is well known [6] that in the above case

$$d_n(\mathscr{E}, l_p) = a_n, \qquad n \geqslant 0$$

so

$$\mathscr{L}_n = \{L, \dim L = n \mid \delta_n(\mathscr{E}; L) = a_n\}.$$

If we let $\{e_i\}_{i=0}^N$ be the coordinate vectors for l_p^N and $H_n = \operatorname{sp}\{e_i\}_{i=0}^{n-1}$, then it is easy to see that H_n is an optimal subspace of dimension n. However, small perturbations of the subspaces also yield optimal subspaces. Below we give a precise description of these perturbations.

For the remainder of this paper, fix $n < N \leqslant \infty$. To simplify notation, it is to be understood that $l_p = l_p^N$.

Set

$$\mathscr{E} = \left\{ x \in l_p : \sum_{i=0}^{N} \left| \frac{x_i}{a_i} \right|^p \leqslant 1 \right\}, \qquad 1 \leqslant p < \infty$$

$$= \left\{ x \in l_\infty : \sup_i \left| \frac{x_i}{a_i} \right| \leqslant 1 \right\}, \qquad p = \infty.$$

Let $\{e_i\}_{i=0}^N$ and $\{e_i'\}_{i=0}^N$ be the coordinate vectors for l_p and l_q, respectively, where $1/p + 1/q = 1$. For each n, with $1 \leqslant n < N$,

$$H_n = \operatorname{sp}\{e_i\}_{i \leqslant n-1} \qquad H^n = \{x \in l_p : x_i = 0, 0 \leqslant i \leqslant n\},$$

$$H_n' = \operatorname{sp}\{e_i'\}_{i \leqslant n-1} \qquad H_n^+ = \{y \in l_q : y_i = 0, 0 \leqslant i \leqslant n\}.$$

On H_n', define

$$|||w|||_1 = \left[\sum_{i=0}^{n-1} (a_i^q - q_n^q) |w_i|^q \right]^{1/q} \qquad 1 \leqslant q < \infty$$

$$= \sup_{0 \leqslant i \leqslant n-1} a_i |w_i| \qquad q = \infty.$$

On H_n^+, define

$$|||y|||_2 = \left[\sum_{i=n+1}^{N} (a_n^q - a_i^q) |y_i|^q \right]^{1/q} \qquad 1 \leqslant q < \infty$$

$$= a_n \sup_{i \geqslant n+1} |y_i| \qquad q = \infty.$$

Since $\{a_i\}_{i=0}^N$is strictly decreasing, it is easy to see that $|||\cdot|||_1$ and $|||\cdot|||_2$ are norms on the spaces H_n' and H_n^+, respectively. For $1 \leqslant p \leqslant \infty$, we have

THEOREM. *Let L be a subspace of l_p*, dim $= n$. *L is an optimal subspace if and only if $L = graph(S)$, where $S: H_n \to H^n$ is a linear operator with* $|||S'y|||_1 \leqslant |||y|||_2 \ \forall\, y \in H_n^+$.

Proof. We give all the details in the case $1 < p < \infty$ and then some comments to adjust this proof to the cases $p = 1$ or ∞. Suppose L is an optimal subspace of dimension n. Then

$$\max_{x \in \mathscr{E}} \operatorname{dist}(x, L) = a_n. \tag{1}$$

Let $Q: l_p \to l_p/L$ be the canonical surjection. $\forall\, x \in l_p$,

$$\|Qx\|_{l_p/L} = \inf_{y \in L} \|x - y\|_p = \operatorname{dist}(x, L).$$

By (1).

$$\|Qx\|_{l_p/L} \leqslant a_n, \qquad \forall\, x \in \mathscr{E}.$$

Define $A: l_p \to l_p$ by $Ae_i = a_i e_i$, $0 \leqslant i \leqslant N$. Thus $\mathscr{E} = A(B)$, where B is the unit ball in l_p. Consider $QA: l_p \to l_{p/L}$

$$\|QA\| = \sup_{x \in B} \|QAx\|_{l_p/L} = \sup_{x \in \mathscr{E}} \|Qx\|_{l_p/L} \leqslant a_n.$$

Hence

$$\|A'Q'\| = \|QA\| \leqslant a_n. \tag{2}$$

Furthermore, $A'|_{L^\perp} = A'Q'$ implies $\|A'|_{L^\perp} \leqslant a_n$.

Next, we assert that $L^\perp \cap H_n' = \{0\}$. If not, then exists some $z \in L_n^\perp \cap H_n'$ with $\|z\|_q = 1$. By (2), $\|A'z\|_q \leqslant a_n$. But $z \in H_n'$, so $\|A'z\|_q \geqslant a_{n-1}$ since $A'z = \sum_{i=0}^{n-1} a_i z_i e_i'$. Thus $a_n \geqslant a_{n-1} > a_n$, a contradiction.

Now let $P_n: L^\perp \to H_{n-1}^+$ be the coordinate-wise projection. $L^\perp \cap H_n' = \{0\}$ implies that P_n and $R_n: H_n' \to l_q/L^\perp$ are 1–1. Since H_n' and $l_q/L^\perp$ are finite dimensional, R_n is an isomorphism. Hence $l_q = L^\perp \oplus l_q/L^\perp$. Using this fact, it is easy to show that P_n is also onto.

For each $y \in H_{n-1}^+$, let $Ty = P_n^{-1} y - y$. Since P_n is a bijection and $P_n^{-1} y = y + Ty$, $L^\perp = \{y + Ty: y \in H_{n-1}^+\}$. Now $\|A'\|_{L^\perp}\| \leqslant a_n$ implies that

$$\|A'y + A'Ty\|_q \leqslant a_n \|y + Ty\|_q \qquad \forall\, y \in H_{n-1}^+$$

by the decomposition of $L^\perp$. So for each $y \in H_{n-1}^+$

$$\|A'y + A'Ty\|_q^q \leqslant a_n^q \|y + Ty\|_q^q. \tag{3}$$

But y and Ty are disjointly supported and the subspaces H'_n and H^+_{n-1} are invariant under A, hence

$$\|A'y\|_q^q + \|A'Ty\|_q^q = \|A'y + A'Ty\|_q^q \leqslant a_n^q \| y\|_q^q + a_n^q \| Ty\|_q^q \tag{4}$$

or

$$\|A'Ty\|_q^q - a_q^q \| Ty\|_q^q \leqslant a_n^q \, y\|_q^q - \|A'y\|_q^q \qquad \forall\, y \in H^+_{n-1}. \tag{5}$$

Substituting e'_n into (5), we have

$$\|A'Te'_n\|_q^q - a_n^q \| Te'_n\|_q^q \leqslant a_n^q \| e'_n\|_q^q - \|a_n e'_n\|_q^q = 0.$$

So

$$\|A'Te'_n\|_q \leqslant a_n \| Te'_n\|_q \qquad \text{and} \qquad a_{n-1} \| Te'_n\|_q \leqslant a_n \| Te'_n\|_q$$

since $Te'_n \in H'_n$. Thus $Te'_n = 0$. Let $T = T|_{H^+_n}$. By (5),

$$\|A'Ty\|_q^q - a_n^q \| Ty\|_q^q \leqslant a_n^q \| y\|_q^q - \|A'y\|_q^q \qquad \forall\, y \in H^+_n.$$

That is,

$$\begin{aligned}
&\sum_{i=0}^{n-1} |(A'Ty)_i|^q - a_n^q \sum_{i=0}^{n-1} |(Ty)_i|^q \\
&\quad \leqslant a_n^q \sum_{i=n+1}^{N} | y_i|^q - \sum_{i=n+1}^{N} |(A'y)_i|^q \\
&\quad \Leftrightarrow \sum_{i=0}^{n-1} a_i^q |(Ty)_i|^q - a_n^q \sum_{i=0}^{n-1} |(Ty)_i|^q \\
&\quad \leqslant a_n^q \sum_{i=n+1}^{N} | y_i|^q - \sum_{i=n+1}^{N} a_i^q | y_i|^q \\
&\quad \Leftrightarrow \sum_{i=0}^{n-1} (a_i^q - a_n^q) |(Ty)_i|^q \leqslant \sum_{i=n+1}^{N} (a_n^q - a_i^q) | y_i|^q \\
&\quad \Leftrightarrow |\!|\!| Ty |\!|\!|_1 \leqslant |\!|\!| y |\!|\!|_2 \qquad \forall\, y \in H^+_n.
\end{aligned} \tag{6}$$

From the decomposition $L^\perp = \{ y + Ty : y \in H^+_{n-1}\}$, it follows that $K_n = \{x - T'x : x \in H_n\} \subseteq L$, where $T' : H_n \to H^+_n$. Given any $x \in H_n$, x and Tx are disjointly supported. Hence $\dim K_n = n = \dim L$ and $L = K_n = \{x - T'x: \quad x \in H_n\}$. Setting $S = -T'$, we have $L = \{x + Sx : x \in H_n\} = \text{graph}(S)$ with $|\!|\!| S'y |\!|\!|_1 \leqslant |\!|\!| y |\!|\!|_2 \ \forall\, y \in H^+_n$.

Since the above steps are reversible, the theorem is proved.

In the case $p = 1$, we obtain the estimates of the norms as in (3)–(6) by observing that $\forall\, y \in H_{n-1}^{+}$,

$$\begin{aligned}
&\|A'y + A'Ty\|_\infty \leqslant a_n \| y + Ty\|_\infty \\
&\quad \Leftrightarrow \sup_i |(A'y + A'Ty)_i| \leqslant a_n \sup_i |(y + Ty)_i| \\
&\quad \Leftrightarrow \max[\sup_{i \geqslant n} a_i | y_i|, \sup_{i \leqslant n-1} a_n |(Ty)_i|] \\
&\quad\ \leqslant a_n \max[\sup_{i \geqslant n} | y_i|, \sup_{i \leqslant n-1} |(Ty)_i|] \\
&\quad \Leftrightarrow \sup_{i \leqslant n-1} a_i |(Ty)_i| \leqslant a_n \sup_{i \geqslant n} | y_i|
\end{aligned}$$

by the nature of the sequence $\{a_i\}_{i=0}^{N}$.

In the case $p = \infty$, let $\{v_1, ..., v_n\}$ be a basis for L. Replace $L^\perp$ by $L^+ = \bigcap_{i=1}^{n} \operatorname{Ker} v_i$. As in the previous cases, $\|A'|_{L^+}\| \leqslant a_n$ and $L^+ \cap H'_n = \{0\}$. The remainder of the proof is the same as in the case $1 < p < \infty$.

Remark. In the above analysis, the *strict* monotonicity of the sequence $\{a_i\}_{i=0}^{N}$ was not necessary. Let $m = \min\{k \leqslant n: a_k = a_{k+1} = \cdots = a_n\}$ and $M = \sup\{k > n: a_n = \cdots = a_{k-2} = \cdots a_{k-1}\}$. Then we have

PROPOSITION. *L is an optimal subspace if and only if $L = \operatorname{graph}(S) \oplus E$, where E is a subspace of $H_M \cap H^{m-1}$ and $S: H_m \to H^{M-1}$ is a linear operator with $|||S'y|||_1 \leqslant ||| y|||_2\ \forall\, y \in H_{M-1}^{+}$.*

The analysis for this case is the same as in the proof of the theorem. If L is an optimal subspace and $Q: L \to H_m \oplus H^{M-1}$ is the coordinate-wise projection, set $E = \operatorname{Ker} Q$. It follows that $(QL)^+$, the coordinate projection of $(QL)^\perp$ into $H'_m \oplus H_{M-1}^{+}$, is isomorphic to H_{M-1}^{+}. (This is an analogue of the statement that $L^\perp \cap H'_n = \{0\}$ and P_n is onto in the case of strict monotonicity—see the two paragraphs before (3).) Thus there is a linear operator $T: H_{M-1}^{+} \to H'_m$ such that $(QL)^+ = \operatorname{graph}(T)$. The remainder of the analysis parallels the arguments in the proof.

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